Michael Todd Edwards and Jeffrey A. Reinhardt

Approximating Irregular Areas with Monte Carlo Simulations

In this month's tips Michael Todd Edwards and Jeffrey A. Reinhardt illustrate the use of Monte Carlo simulations to explore areas of irregular regions bounded by two or more functions. Tip 1 illustrates how to perform such tasks with the TI-84 graphing calculator. The second tip extends the Monte Carlo investigation with the dynamic statistics software Fathom by Key Curriculum Press. The "Surfing Note" this month includes links to sites with more information about Monte Carlo methods.

TIP 1: USING A TI-84 GRAPHING CALCULATOR

In introductory high school courses, students typically spend significant time calculating areas of circles, triangles, and various quadrilaterals (e.g., rectangles, trapezoids, kites). They spend far *less* time calculating areas of irregular shapes. This allocation of effort is unfortunate, because many "real-world" objects are clearly irregular (e.g., political and geographic regions, dress patterns, oil spills, etc.) and because such calculations are accessible to students using technology.

Consider the region bounded by the curve $f(x) = 8x^2$ and the lines $x = 0$, $x = 0.5$, $y = 0$, and $y = 2$. A graph of this region shown in **figure 1** may be generated using the window settings Xmin=0, Xmax=0.5, Xscl=1, Ymin=0, Ymax=2, Yscl=1, and Xres=1.

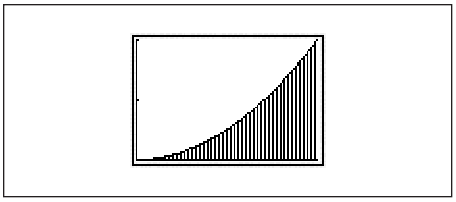

Fig. 1 Region bounded by $f(x) = 8x^2$, $x = 0$, $x = 0.5$, $y = 0$, and $y = 2$

Although the area of the shaded region may be determined using integral calculus, students in introductory courses may approximate the area using Monte Carlo simulations. In a Monte Carlo study, a problem or experiment is modeled, and random (or pseudo-random) numbers are generated to imitate chance behavior based on this model. After repeated trials, a statistical analysis of the data can be performed. In this example, students use graphing calculators to select random ordered pairs (x, y) within the rectangular region bounded by lines $x = 0$, $x = 0.5$, $y = 0$, and $y = 2$. This region is labeled as *ABCD* in **figure 2**.

Note that the probability that a randomly selected ordered pair (x, y) will lie in the shaded region equals the ratio of the shaded area to the area of rectangle *ABCD*. In other words,

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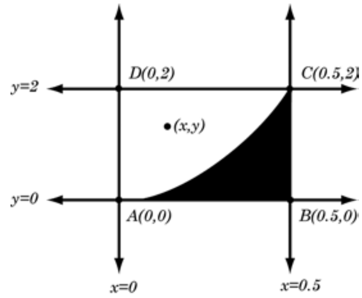


Fig. 2 $f(x) = 8x^2$ in the rectangle formed by $x = 0$, $x = 0.5$, $y = 0$, and $y = 2$

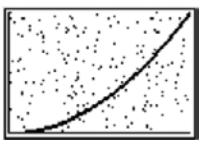


Fig. 3 Scatter plot of 200 random ordered pairs

$$P[(x, y) \text{ lies in shaded region}] = \frac{\text{Area of Shaded Region}}{\text{Area of } ABCD}.$$

From this result, it follows that

$$\text{Area of Shaded Region} = P[(x, y) \text{ lies in shaded region}] \times \left(\text{Area of } ABCD\right).$$

Students can use graphing calculators to simulate the random selection of a relatively large number of ordered pairs, say 200, that lie within region *ABCD.* The *x*- and *y*-coordinates of the random points are stored in lists L1 and L2, respectively. Students define lists L1 and L2 in the list editor of their graphing calculators using the following commands:

L1=0.5*rand(200)
L2=2*rand(200)

Note that the expression rand(200) returns a list of 200 random numbers between 0 and 1. Hence, the command L1=0.5*rand(200) places 200 random numbers with values between 0 and 0.5 in list L1. Similarly, the command L2=2*rand(200) places 200 random numbers with values ranging between 0 and 2 in list L2.

Next, students graph the curve $f(x) = 8x^2$ along with a scatter plot of the (x, y) coordinate pairs stored in lists L1 and L2

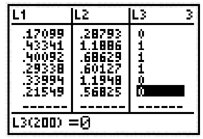


Fig. 4 L3 indicates whether the random pair lies in the region.

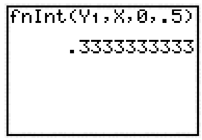


Fig. 5 Finding the area by using the calculator's built-in function

(see **fig. 3**). Students can determine if a particular ordered pair falls within the irregular region by defining list L3 as L2<8*L1^2. **Figure 4** illustrates the results of defining such a list.

In the list editor, the TI-84 represents "true" as "1" and "false" as "0." Thus, when corresponding values in L1 and L2 make the expression L2<8*L1^2 true, a "1" is displayed in L3. If the expression is false, a "0" is displayed. Graphically speaking, a "1" appears in list L3 when corresponding values in L1 and L2 generate an ordered pair that lies *within* the irregular region. A "0" appears when corresponding values in L1 and L2 generate an ordered pair that lies *outside* the irregular region.

Summing L3 will indicate how many points lie within the irregular region; dividing this sum by 200 leads to an approximation for the area of the irregular region.

At this point, students compare results with one another and discuss why their approximations vary. A class average approximation can then be calculated. The effects of increasing the number trials might be investigated. Results could also be compared with the value of the area under the curve $y = 8x^2$ for x ranging from 0 to 0.5 using integral calculus or calculator-based integration such as that shown in **figure 5**.

TIP 2: USING FATHOM, VERSION 2

Fathom provides students with powerful, easy-to-use data analysis tools in a dynamically linked, multi-representational environment. Students can use Fathom to approximate the area of the region bounded by the curves $y = x^2 - 10x + 28$ and $y = -x^2 + 10x - 18$ by conducting a Monte Carlo simulation.

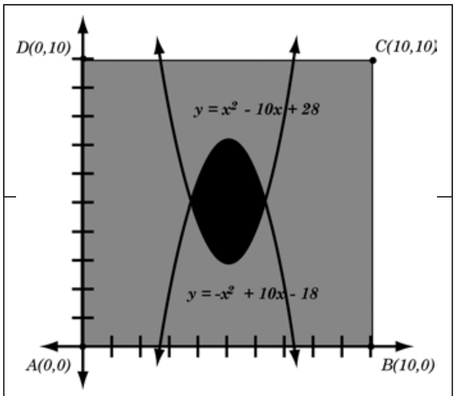


Fig. 6 Monte Carlo set up for determining a second irregular area

First, recall that the

$$\text{Area of Shaded Region} = P[(x, y) \text{ lies in shaded region}] \times \left(\text{Area of } ABCD\right).$$

In the Monte Carlo simulation, the data are a collection of points that lie within the square *ABCD,* as shown in **figure 6**. Each point is considered a separate case within the collection of all points. The attributes of each point include its *x*- and *y*-coordinates. Therefore, students must define attributes for this collection: *x, y,* and in_region, a boolean expression that indicates whether the ordered pair lies in the region bounded by curves $y = x^2 - 10x + 28$ and $y = -x^2 + 10x - 18$.

Since the ordered pairs in the simulation are randomly selected, the x and y attributes can be defined with Fathom's random function (see **fig. 7**).

Although attributes for each ordered pair in the collection have been defined, the collection itself contains no actual ordered pairs. The 0/0 field at the bottom left corner of the inspection window indicates this (see **fig. 7**).

Next, students must instruct Fathom to simulate the random selection of 1000 ordered pairs within square *ABCD.* Hence, they will need to add 1000 cases to the collection of ordered pairs. To add ordered pairs to the collection of data, first select the collection, then select the New Cases option underneath the Collection menu item, and specify the number of cases to add in the Add Cases dialogue box.

As **figure 8** highlights, the addition of 1000 cases is reflected by the "1/1000" field at the bottom left corner of the Inspect Collection window. Clicking on the arrows in the bottom left corner of the window enables students to browse x- and y-coordinates of various points in the collection.

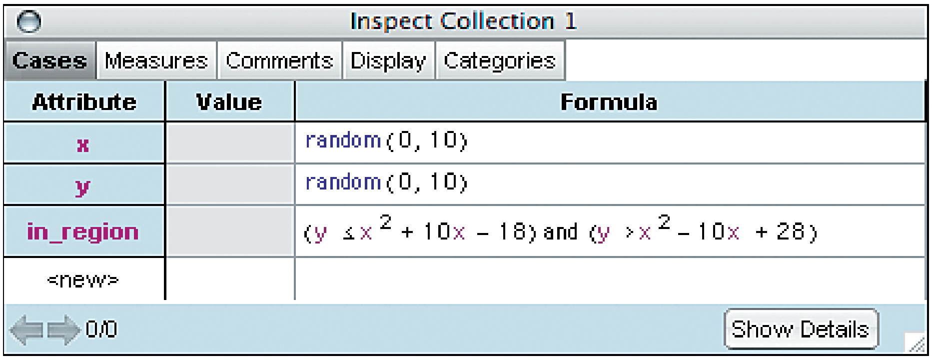


Fig. 7 Formulas for *x*, *y*, and in_region

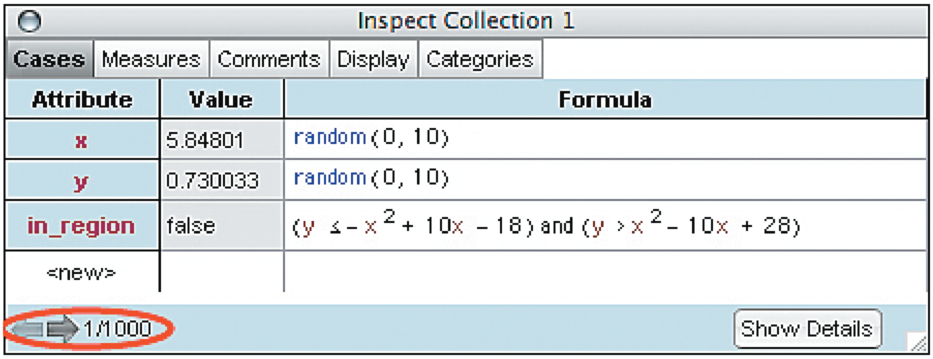


Fig. 8 The "Inspect Collection" window indicates that 1000 cases exist.

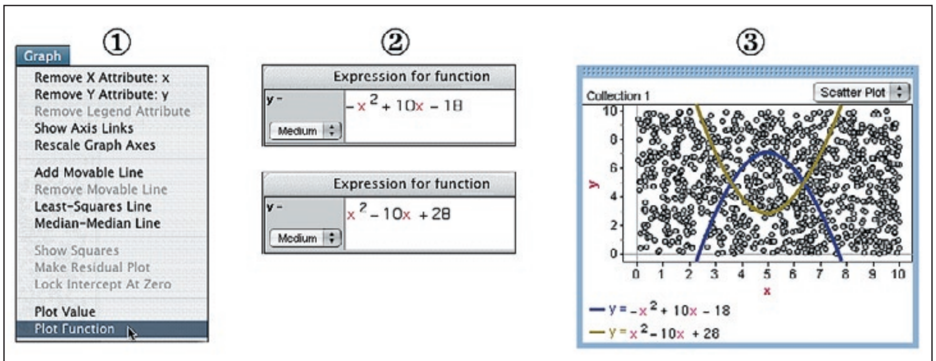


Fig. 9 Randomly generated ordered pairs and graphs of $y = x^2 - 10x + 28$ and $y = -x^2 + 10x - 18$

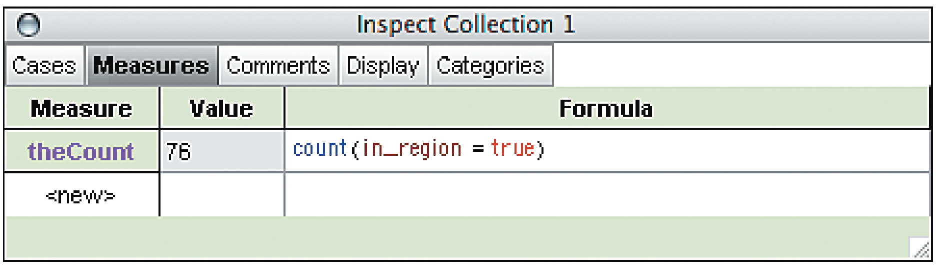


Fig. 10 Defining a measure in Fathom

Rather than perusing x- and y-coordinates of the randomly generated ordered pairs individually, students can construct a scatter plot to display all ordered pairs simultaneously. To obtain the scatter plot, select the attribute x in the collection and drag it to the horizontal axis of the plot. Drag attribute y to the vertical axis. The curves $y = -x^2 + 10x - 18$ and $y = x^2 - 10x + 28$ can then be added to the scatter plot graph. To do this, click on the graph icon and move the cursor to the workspace to create an empty plot. Change the empty plot to a function plot and choose the Plot Function option from the Graph menu item, then individually enter the functions to be graphed. The results of these steps are shown in **figure 9**.

A measure can then be defined in the collection inspector to count the number of ordered pairs contained within the irregular region bounded by the curves $y = x^2 - 10x + 28$ and $y = -x^2 + 10x - 18$. To define a measure, click on the Measures tab within the Inspect Collection window, then type the definition into the corresponding Formula cell. **Figure 10** highlights a measure of the number of ordered pairs that lie within the region bounded by the graphs of $y = x^2 - 10x + 28$ and $y = -x^2 + 10x - 18$. In this case, the count function counts the number of cases with a true in_region attribute.

As indicated in **figure 10**, 76 of the 1000 ordered pairs lie within the irregular region. This result leads to the following approximation of the area for the irregular region:

$$\begin{aligned}&\text{Area of Shaded Region}\\ &\quad = P[(x,y) \text{ lies in shaded region}]\\ &\qquad \times \left(\text{Area of } ABCD\right)\\ &\quad \approx \frac{76}{1000} \cdot 100 \text{ square units}\\ &\quad = 7.6 \text{ square units}\end{aligned}$$

By re-randomizing cases within the collection (by typing Ctrl-Y), students may rerun the simulation, obtaining a different approximation of the area of the irregular region.

More powerfully, students can instruct Fathom to store results of numerous runs automatically by storing measures from the collection. To do this, highlight the collection object in the Fathom document window. Select the Collect Measures item from the Collection menu. A new Measures from Collection icon appears in the document window. Initially, this container holds 5 "theCount" measures from 5 separate runs of the Monte Carlo simulation. Double click on the Measures from Col-

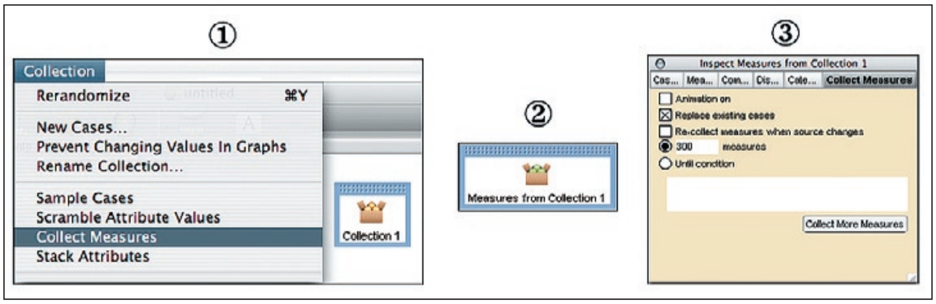


Fig. 11 (1) Select "Collect Measures" option; (2) Double click on the "Measures from Collection" icon; (3) Specify number of measures to collect.

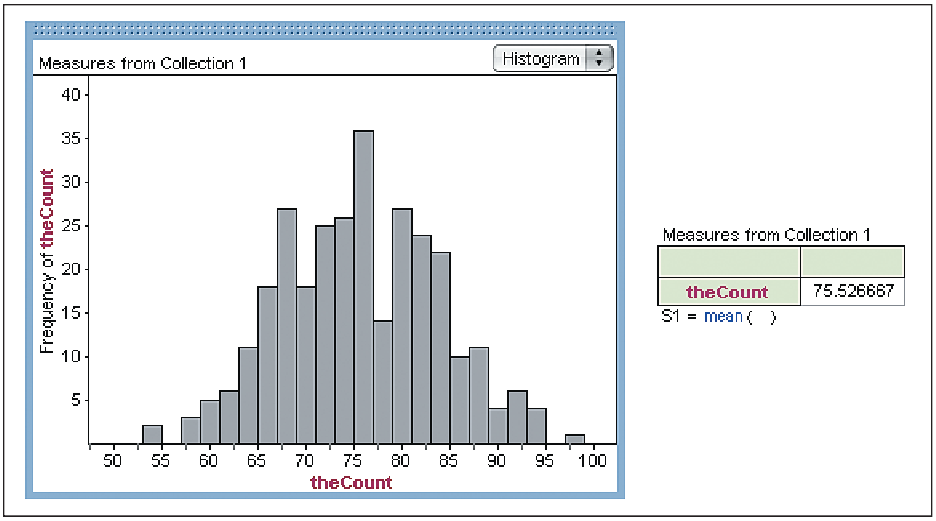


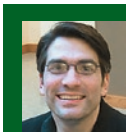
Fig. 12 (a) Histogram illustrating the frequency of various numbers of ordered pairs within the irregular region for 300 runs of the simulation; (b) The mean number of ordered pairs within the irregular region.

lection icon. Underneath the Collect Measures tab, specify that you would like to collect 300 measures. Then click the Collect More Measures button. These steps are displayed in **figure 11**.

Click on the Cases tab within the Inspect Measures window. By dragging the theCount case attribute to an empty graph, a histogram of the frequency of various numbers of ordered pairs within the irregular region for 300 runs of the simulation may be generated (see **fig. 12**). (Collection time will be markedly less if Animation is turned off.)

The mean number of ordered pairs within the irregular region for the 300 runs equaled 75.526667. To obtain this mean, define a measure from the appropriate tab in the Measures from Collection window. The mean command is an option in the One Attribute folder under Statistical in the list of commands available when using the Calculator window. From these results, one may approximate the area of the irregular region as 7.552 square units.

As shown in Tips 1 and 2, Fathom or the TI-84 graphing calculator can be employed to approximate irregular areas with Monte Carlo simulations.

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Surfing Note

polymer.bu.edu/java/java/montepi/MontePi.html

Paul Trunfio, a research associate at the Center for Polymer Studies and Department of Physics at Boston University, with the help of Gary McGrath, created a Java applet of the Monte Carlo estimation for π. The applet simulates randomly throwing 1, 100, 1,000, or 10,000 darts at a circle inscribed in a square while estimating "the ratio of the area of the circle to the area of the square, by counting the number of darts in each." Also available at this Web site is a link to a Java applet of the Buffon Needle estimation for π.

random.mat.sbg.ac.at/links/monte.html

At The WWW Virtual Library: Random Numbers and Monte Carlo Methods, readers can learn more about the theory behind, applications of, software for, and conferences about Monte Carlo methods. One particular software link describes how the Crystal Ball Standard Edition, an add-in product for Excel, uses the Monte Carlo simulation to create thousands of possible alternative outcomes in the modeling of risk management in oil and gas exploration. Learn more about this by visiting

www.decisioneering.com/oilandgas/index.html ∞